My Solutions to Old Analysis Quals

Jacob S Townson

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# Prelude

This document is written in reference to qualifying exams given at the University of Louisville in past years. These solutions are not given from the University, but of my work alone as a way to study for my own qualifying exam. If any tips or recommendations come up and you feel you should share, feel free to raise an issue on GitHub where I have this document saved and open to the public [here](https://github.com/obewanjacobi/gradwork/tree/master/Classes/Old%20Qualifying%20Exams/My%20Solutions). To see the qualifying exams for yourself, [visit this link](http://www.math.louisville.edu/GraduateFAQ/qualifiers/QualifierStudyGuides/). When referencing the Royden book, this is in reference to the *Real Analysis; 4th Edition*. Special thanks to Trevor Leach for sharing his solutions with me for reference on this document. Thank you for reading, and for every advice.

Jacob Townson

# What to Expect:

* differentiation and Riemann integration of functions of one real variable, sequences of functions, uniform convergence, Lebesgue’s characterization of Riemann integrability
* topology of the line, countable and uncountable sets, Borel sets, Cantor sets and Cantor functions, Baire category theorem
* Lebesgue measure and integration on the line, measurable functions, convergence theorems
* AC and BV functions, fundamental theorem of calculus, Lebesgue differentiation theorem
* Hilbert spaces, Lp spaces, lp spaces, Hölder and Minkowski inequalities, completeness

# My Solutions from Quals

## Spring 2017

### Prove that if has Lebesgue measure , then .

#### My Solution:

Let and . First note that we know is continuous on . Let , then

Also, for all . By definition of Lebesgue measure, for all , there exists a sequence of intervals such that is the empty set for all and . Also, such that since . Let , , then . Then

Since is continuous, we know that . Since is a closed interval and is continuous, we can find and such that and . WLOG we can assume that . Then,

where

Since and , we find that . This implies which implies

Thus, . QED

### Prove that if is absolutely continuous, then so is .

#### My Solution:

Given that is absolutely continuous, we know then that for all , there exists a such that if is a finite collection of disjoint intervals of with implies that . Also, because is absolutely continuous, we know then that it must also be continuous. Hence, must be bounded on . Let on and choose such that gives us

Then note that

Hence, is absolutely continuous as well. QED

### Prove that if is absolutely continuous on and there is a such that a.e., then is differentiable on and .

#### My Solution:

Using Lebesgue’s fundamental theorem of calculus, we know that

Then, because a.e.;

This implies that , thus by the fundamental theorem of calculus, if we take the derivative of both sides, we can see that for all . Note, the reason that we can use the fundamental theorem of calculus here is because the function is absolutely continuous, which implies it’s of bounded variation. QED

### Prove that the series converges uniformly for and then evaluate the following series:

#### My Solution:

Recall the **Weierstrass M Test**: Let be a series of real valued functions on a subset . Suppose there exists a convergent series where such that for all and , . Then converges uniformly.

Now, for , for all . Note, converges because it is a geometric series. Thus by the Weierstrass M test, the series converges uniformly for . QED

Because the series converges uniformly on a compact set,

QED

### Let be a sequence of Lebesgue measurable subsets of . Prove: (a) If then ; and (b) If it may not be true that

#### My Solution:

1. Let , since , there exists an such that . So,
2. Hence,
3. This implies that .
4. Let , , , , , , , ,… We repeat these definitions in this pattern, giving us that . But for all , implying that . Thus it may not be true that if that . QED

### Prove that if , , then

#### My Solution:

Let and . Note, since , (because , , which implies that must be finite). So,

where

and

since . Thus by the Lebesgue Dominated Convergence Theorem,

QED

### Define the function by if is irrational, and by if is rational and when written in least terms. Decide whether or not is Riemann integrable on and if so, evaluate its integral.

#### My Solution:

Let , then . Thus . Also is clearly measurable as . Hence is Lebesgue measurable and with since a.e. QED

### Let be a sequence of polynomials. Suppose that for every point there exists an index satisfying . Prove at least one of the polynomials is identically zero.

#### My Solution:

Suppose there does not exist an such that for all . Let’s define . Note, has to be finite since each polynomial can only have a finite number of zeros. Now consider since for each there exists a polynomial such that . But a countable collection of finite sets is countable. But is uncountable, thus giving us a contradiction. This implies that one of the polynomials must indeed be identically zero. QED

### Let . Prove that the following are equivalent to each other: (a) is not Lebesgue measurable; and (b) There is an such that whenever is measurable and , then .

#### My Solution:

Let not be Lebesgue measurable and suppose that for all there exists such that and . So for every , let and . Then and . Since which implies is measurable (because Lebesgue measure is complete). Hence, is measurable. This is a contradiction, thus we have proven the desired result. QED

### Let . Define a functional by . Prove that

#### My Solution:

* Will show :

Note, by Holder’s since we have ,

* Will show :

We may assume . is -finite, so there exists increasing towards such that . Define for , to be fixed. Hence . Define , which is implies for all and for all . Thus for all . This implies .

Thus we have proven the desired result. QED

## August 2016

### Let be a closed set. Prove that is Riemann integrable iff has Lebesgue measure zero.

#### My Solution:

Assume that has Lesbegue mesure zero. This implies that . Thus the Riemann integral exists and agrees with the Lesbesgue integral.

Now assume that is Riemann integrable. This is true iff the Lesbegue integral exists and . So is discontinuous at its boundary points, which implies that . QED

### Let Prove the following statements are equivalent: (a) is Lebesgue measurable and (b) There is a set and a set of measure zero such that .

#### My Solution:

(): Given is a set, must then be Borel measurable. Thus it is also Lebesgue measurable. must be Lebesgue measurable as well since the Lebesgue -algebra is complete. Thus is Lebesgue measurable.

(): This follows directly from a proposition stating that if is a Lebesgue measurable set, and is a Lebesgue measure, then there exists a set which cintains that is the countable intersection of a decreasing sequence of open sets and . QED

### If is nonnegative and integrable on , then

#### My Solution:

where . Thus,

Since is integrable, we know that , thus is integrable as well. So,

by the D.C.T., and thus,

QED

### Let . If for all rational numbers and with , then a.e.

#### My Solution:

We will claim here that integrates to over arbitrary open sets. Thus for any , choose an open set such that and . Hence

because . Thus the integral is since this is true for all .

We must now prove our claim in order to complete the problem. For any , there exists such that decreases to and increases to as goes to infinity. Thus . Using this, and the dominated convergence theorem, we find that

with as the majorant since . Now let with being arbitrary disjoint intervals. Then

By the dominated convergence theorem,

with as the majorant because . Thus for any open set . A similar proof holds for . Hence a.e. QED

### Suppose that and . Define . Prove that and that .

#### My Solution:

For , we have

So . Given this, we can see that

by Holders’ because and are conjugate. Then,

QED

### Let be a sequence of real numbers with the property that for all . Prove that both series , converge uniformly on every compact subinterval of and that for all .

#### My Solution:

Since ,

which converges uniformly because by the Weirstrass-M test (and because it’s a geometric series). Let be a compact interval of and let . Then which converges because it’s a geometric sequence. Now we apply the Weirstrass M-test to give us the desired result.

As for proving that , we simply note that by definition of differentiation on power series that this is true (with simple calculus II logic). QED

### Give an example of a continuous function with the property that , , yet for almost every .

#### My Solution:

Let be the Cantor function on . Consider . Then and . Also, almost everywhere the derivative of the Cantor function is 0 (because almost everywhere it is constant). Then the derivative almost everywhere of would be almost everywhere.

### Let be a sequence of measurable subsets of with the property that . Show almost every is contained in only finitely many of the .

#### My Solution:

By way of contradiction, suppose there exists an such that . Also assume that for all where is a subsequence of . But

which diverges because . Thus by contradiction, we have proven the desired result. QED

### Let be Lebesgue measurable. Prove that if for all , then exists and .

#### My Solution:

First off, it is known that . Since , and since we’re integrating between and , we can easily see then that . Let , then which implies that . Thus is integrable. QED

### Define a sequence of functions by if and if . Does converge in ? If so, to what function?

#### My Solution:

Suppose in . Then . But for all . Hence . However, this is a contradiction because , thus it does not converge in . QED

## January 2016

### Let be dense in and . Prove or give a counterexample: is measurable iff is measurable for all .

#### My Solution:

First assume is measurable. This implies that is Lebesgue measurable for all . Hence is Lebesgue measurable for all .

Now assume that is measurable for all . It suffices to show that is measurable for all . So, let , given that is dense in . Let such that decreases to . Thus,

is the union of measurable sets. Hence is measurable. QED

### Suppose denotes the Lebesgue measure of the set . Let be absolutely continuous and be such that . Prove that .

#### My Solution:

Given with , let be a collection of disjoint intervals covering with . is absolutely continuous, which implies that it must also be continuous. Thus for each , let such that . Also, recall that because is continuous, this implies that for all , there exists a such that where . Now notice that

because . Thus

because . Thus . QED

### Let for each . Prove that the sequence converges uniformly on for each , and converges non-uniformly on .

#### My Solution:

Before we begin, note that if , then

. This will be useful in our proof.

Let and . Also let . Hence for , which implies that . The reason that the inequality changes directions here is because for and for very small (specifically less than ), . Thus, we can see that , which implies that for all . Thus converges uniformly on .

If we fix , this implies that by our previous point in the proof. However, it is not uniform since depends on . Thus we have proven the desired result. QED

### Let denote the Lebesgue measure of the set . Find an open set which is dense in such that and for any interval .

#### My Solution:

Let represent the rationals in . For each define as an interval containing , and . Let with

Since contains the rationals on , it is also dense on . is open because it is the countable union of open intervals on . let . It must contain a rational, which implies that it intersects any nontrivially, thus . Thus is a dense open set on such that the measure of is less than and intersects any interval of nontrivially. QED

### Is separable, where ?

#### My Solution:

Yes! First note that seperable means that it contains a countable number of dense subsets.

Let be step functions on and let be step functions of with rational endpoints. Since is dense in , is dense in thus it’s dense in . QED

### Suppose that and that . Prove that if in and in , then in .

#### My Solution:

Here it is sufficient to show that . Well,

by Holder’s Inequality, where , , and . Thus we have proven the desired result. QED

### Assume that . Prove that for each and that .

#### My Solution:

To complete this proof, we will divide the problem into a group of lemmas and prove them to get the desired result.

**Lemma 1**: *Proof*: Assume . Then a.e. Then . Thus .

**Lemma 2**: Assume . Then for . *Proof*: Consider .

* where for , since .
* where . Thus , which implies .

**Lemma 3**: *Proof*: a.e. This implies that , which implies . Hence .

**Lemma 4**: *Proof*: Let . Then . Then . Hence . This implies that . Thus for all . It follows that by our claims then that ||f||*p ||f||*{} ||f||\_p$. Thus we have proven the desired result. QED

## August 2015

### Let denote the Cantor set. Let if and otherwise. Explain why is Riemann integrable and compute .

#### My Solution:

Given that , a.e. which implies that and since is bounded with . Thus . QED

### Let be a Lebesgue measurable set with . Prove there exists a Lebesgue measurable set with .

#### My Solution:

. This implies that . Pick a sufficiently large such that . Consider for . Since is continuous with and , there exists such that by the Intermediate Value Theorem. Thus is a Lebesgue measurable set with . QED

### Let be a measure space. If is a sequence of functions such that converges, then prove that almost everywhere.

#### My Solution:

by the M.C.T. since is increasing. Thus almost everywhere because it’s integrable when approachest . Hence almost everywhere. QED

### Prove that if and if is continuous but not absolutely continuous on .

#### My Solution:

Consider , which is constructed of functions that are continuous everywhere on their domains. So it is continuous on excluding . To show continuity at , consider that which implies that with Hence by the squeeze theorem, .

Suppose is absolutely continuous on . Hence it is of bounded-variation on , thus . Consider the partition with endpoints

Hence for ,

because is even and is odd. The above sum is equal to which diverges because it’s a harmonic series. Thus . This is a contradiction to the assumption of being absolutely continuous, thus we have proven the desired result. QED

### Let be a finite measure space. If is -measurable and for all , then prove that exists and .

#### My Solution:

Since is finite, and , . This implies that .

Now, we must show that is integrable. Consider . Hence . This implies that . Hence is integrable. QED

### Suppose that and that . Prove that if in and in , then in .

#### My Solution:

It is sufficient to show that the . Well,

by Holder’s Inequality where . Thus and . So the above limit does in fact imply that in . QED

### Evaluate . Justify your computations.

#### My Solution:

Here we will use Leibniz’s integral rule, stating that

Thus we find that

where . QED

### If and , then prove that .

#### My Solution:

Given . This implies that and . So,

For , . Thus

where and thus the entire equation is less than . So for all . Thus . QED

### If is measurable, then .

#### My Solution:

Since is positive, is a sequence of increasing positive functions. Thus

with the step of moving the limit inside the integral is by the Monotone Convergence Theorem. QED

## January 2013

### Show that every dense subset of is uncountable.

#### My Solution:

Let . Then for any such that . So,

Without loss of generality, , then on respectively. This is where we get the above equality, as

Consider which is an uncountable collection of disjoing balls. And given any dense set has elements in each ball by definition of density. Thus is uncountable. QED

### Let be a Lebesgue measurable function on with the property that . Prove that and .

#### My Solution:

For every , let

and let . Also define the linear functional such that

Clearly is a bounded functional in , since, by Holder’s inequality,

Since

we can thus conclude that . Moreover, increases to as . So

By the uniform boundedness principle, we can conclude that the sequence converges to a bounded linear functional and that

On the other hand, by the monotone convergence theorem,

hence . Finally, taking in the assumption, we find that

Thus we have proven the desired result. QED

### Let and for all . If for all , then there is a set such that a.e.

#### My Solution:

We know for all . Suppose . Pick such that . Then

since .

Cosider , which we claim is equal to zero. To prove this, pick . Then

as . Hence or almost everywhere. If we use this to define , then almost everywhere. QED

### If is a measurable subset of , then there is an interval such that or .

#### My Solution:

Suppose not! Then for all , and . Suppose has finite measure and let . Then

Thus for all covers of , . But, by definition

Hence

Now for with any measure,

Hence for all where

Note the same proof works for . Thus the desired result is proven. QED

### A measure space is -finite iff there is an such that .

#### My Solution:

Assume that such that . Then

where

since and by Chebyshev’s Inequality.

Now assume is -finite. We have , a union of disjoint sets wuth . Define where when and when . Then

Thus proving the desired result. QED

### (a) Find a sequence such that for all and for all . (b) If the are as in part (a), then prove .

#### My Solution:

1. $f\_n(x):=4n^2x+1 \text{ for }x\in \left[0,\frac{1}{2n}\right]\\ f\_n(x):=-4n^2x+1+4n \text{ for }x\in \left[\frac{1}{2n},\frac{1}{n}\right]\text{and }f(x)=1\text{ otherwise}.$
2. Given as defined above, we can simply integrate to get this answer. So, given defined as in part (a),

$$f\_n(x) -1 =4n^2x \text{ for }x\in \left[0,\frac{1}{2n}\right]\\
f\_n(x)-1=-4n^2x+4n \text{ for }x\in \left[\frac{1}{2n},\frac{1}{n}\right]\text{and }f(x)=0\text{ otherwise}.$$

1. Thus

$$\lim\_{n \to \infty} \int\_0 ^1 |f\_n -1| dx = \lim\_{n \to \infty} \left[ \int\_0 ^{1/2n} 4n^2x dx + \int\_{1/2n} ^ {1/n} (-4n^2x+4n)dx \right]\\
= \lim\_{n \to \infty} \left[ \frac{2n^2}{4n^2} -0 - \frac{2n^2}{n^2} + \frac{4n}{n} + \frac{2n^2}{4n^2} - \frac{4n}{2n} \right] = \lim\_{n \to \infty} 1 = 1$$

1. QED

### Show that is open in . (Assume has the uniform metric.)

#### My Solution:

The function is continuous. Thus must be open. QED

### Let be a metric space and suppose and are nonempty disjoint subsets of with compact and closed. (a) Prove there is a such that for all and . (b) Show that part (a) may fail if is closed, but not compact.

#### My Solution:

1. By way of contradiction, assume for every we can find an and such that . Since is compact, there is a convergent subsequence whose limit is . By the triangle inequality, . If we juggle the ’s around a bit, we find that giving us a contradiction, thus we have proven the desired result. The problem here is that this question is not written that well in regards to our qualifying exam, so for the most part this one should be ignored. It leaves out important details of what exactly is. However, if you would like to check out other solutions on this, follow these links:

[<https://math.stackexchange.com/questions/185656/show-that-exists-delta-0-such-thatdx-y-geq-delta>]

[<http://www.math.ucsd.edu/~benchow/F16/HW7-140A-F16-ans.pdf>] (check #8)

1. Consider an example in with the standard metric. Take as the -axis and as the graph of the expnential function , that is, . These are clearly non-empty and mutually disjoint. Both and are closed, but neither is compace because they are both unbounded. It is easy then to see that their distance is zero and a stricly positive of the desired result does not exist. Thus we can see then that if is not compact and still closed that our proof in part (a) may not hold.

### The limit superior of a sequence of sets is defined as . Let be a sequence of sets in . (a) Prove that if , then . (b) Is it true in general that ?

#### My Solution:

(a): Let . Given , there exists such that .

Hence

which implies that . QED

(b): Consider the sequence of functions: , and so on. Then but for all , . This implies that . Thus it may not be true that . QED

### Show that where and where is in , but where and where is not.

#### My Solution:

Because , it is sufficient to show this is true for as follows from it. Well,

Then if we let and , then

Thus the total variation is finite, thus proving that is of bounded variation.

Now we must show that is not of bounded variation.

Well, let

. Hence for ,

$$\sum^{\infty} \_{n=1} |f(x\_n) - f(x\_{n+1})| = \sum^{\infty} \_{n=1} \left| f \left(\sqrt{\frac{1}{n \pi}} \right) - f \left( \sqrt{\frac{1}{(n+1) \pi}} \right) \right|\\
=\sum^{\infty} \_{n=1} \left| \left( \frac{1}{n \pi} \sin(n \pi) - \frac{1}{n \pi + \pi} \sin(n \pi + \pi) \right) \right|\\
=\sum^{\infty} \_{n=1} \left| \left( \frac{1}{(n+1) \pi}+ \frac{1}{n \pi} \right) \sin(n \pi) \right| \leq \frac{1}{\pi} \sum^{\infty} \_{n=1}\left| \frac{2n+1}{n^2+n} \right|$$

Notice that this is a harmonic series, thus it diverges, implying that , thus it cannot be of bounded variation. QED

# Extra Problems

# Important Notes

## Undergrad

For reference, these notes are gathered from the book *Real Analysis; A First Course* by Russell A. Gordon. These notes consist of basic real analysis ideas based off of my past undergraduate class taught by Dr. Christine Leverenz at Georgetown College.

### Gordon Chapter 1 (Real Numbers)

* A **field** is a nonempty set of objects that has two operations defined on it. These operations are usually defined as addition and multiplication. These operations follow a set of properties which will not be listed here as you should know them.
* **Triangle Inequality**: . It follows from this that
* If and are real numbers, then
* **Cauchy-Schwarz Inequality**: Let be a positive integer. If and are real numbers, then
* Equality occurs iff there is a constant such that for all integers
* The set is **bounded** if there is a number such that for all . The number is called a **bound** for S.
* Suppose that is bounded above. A number is the **supremum** of if is an upper bound of and any number less than is not an upper bound of . We write
* Suppose that is bounded below. A number is the **infimum** of if is an lower bound of and any number greater than is not an lower bound of . We write
* **Archimedean Property of the Real Numbers**: If and are positive real numbers, then there exists a positive integer such that .
* Between any two distinct real numbers, there exists a rational and an irrational number.
* A countable union of countable sets is countable.
* Let be an interval and be a function such that , and let be a subinterval of . The function is **increasing** on if for all such that ; and **strictly increasing** on if for all such that . The function is **decreasing** on if for all that satisfy ; and **strictly decreasing** on if for all such that . The function is **monotone** on if it is either increasing or decreasing on and **strictly monotone** on if it is either strictly increasing or decreasing on .

### Gordon Chapter 2 (Sequences)

* A **sequence** is a function whose domain is the set of positive integers. A **sequence of real numbers** is a sequence whose codomain is the set . Although a sequence is a function, the standard notation for a sequence of real numbers is where the subscript denotes the **index** of the sequence.
* A sequence **converges to a number** if for all there exists a positive integer such that for all . The sequence is **convergent** if there exists a number that the sequence converges to, otherwise it is **divergent**.
* The limit of a convergent sequence is unique.
* Suppose that converges to and converges to . Then:

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* A monotone sequence converges iff it is bounded
* A sequence is a **Cauchy Sequence** if for all there exists a positive integer such that for all .
* A sequence of real numbers converges iff it is a Cauchy sequence
* Let be a sequence and let be a strictly increasing sequence of positive integers. The sequence is called a **subsequence** of
* If a sequence converges to , then every subsequence must converge to as well.
* **Bolzano-Weierstrass Theorem**: Every bounded sequence has a convergent subsequence.

### Gordon Chapter 3 (Limits and Continuity)

* Let be an open interval that contains the point and suppose that is a function that is defined on except possibly at . The function has **limit** at point if for all there exists such that for all such that . We then write .
* We have linearity for limits.
* Let be an interval and , and let . The function is **continuous** at if for each there exists a such that for all such that . The function is continuous on if is continuous on every point of .
* **Intermediate Value Theorem**: Suppose that is continuous on . If is a number between and , then there is a point such that .
* **Extreme Value Theorem**: If is continuous on , then there exist points and in such that for all .
* Let be an interval. A function is **uniformly continuous** on if for each there exists a such that for all such that .
* If is continuous on , then is uniformly continuous on .
* A **partition** of an interval is a finite set of points such that
* Let be a function and let be any closed subinterval of . The **variation** of on is defined by . Note that the integer is not fixed; the supremum is over all possible partitions of . The function is of **bounded variation** on if is finite.

### Gordon Chapter 4 (Differentiation)

* Let be an interval, let , and let . The function id **differentiable** at provided that the limit
* exists. The **derivative** of at is the value of the prementioned limit denoted by .
* **Rolle’s Theorem**: Let be continuous on and differentiable on . If , then there exists a point such that .
* **Mean Value Theorem**: If is continuous on and differentiable on , then there exists a point such that

### Gordon Chapter 5 (Integration)

* A **tagged partition** of an interval consists of a partition of along with a set of points, known as **tags**, that satisfy for .
* Let and let be a tagged partition of . The **Riemann sum** of associated with is defined by
* A function is **Riemann integrable** on if there exists a number with the following property: for all there exists such that for all tagged partitions of that satisfy . The number is called the **Riemann integral** of on .
* **Cauchy Criterion for Riemann Integrability**: A bounded function is Riemann integrable on iff for each there exists such that for all tagged partitions and of with norms less than .
* Fundamental Theorem of Calculus is a thing
* **Integration by Parts**:

### Gordon Chapter 6 (Infinite Series)

* A **power series** is an expression of the form
* where the ’s are constants
* A **Fourier series** is an expression of the form
* where the ’s and ’s are constants
* An **infinite series** of real numbers is an expression of the form
* A **partial sum** of an infinite series is represented by
* An infinite series **converges** if its corresponding sequence of partial sums converges. If is the limit of the previous sequence, then we say the series converges to . If the sequence does not converge, we say that the series **diverges**.
* If the series converges, then the sequence converges to zero.
* The series converges iff for all there exists a positive integer such that for all positive integers and that satisfy
* A series with nonnegative terms converges iff its sequence of partial sums is bounded
* Linearity is preserved
* **Geometric Series**: Suppose that . The geometric series converges if and diverges if . If ,
* The -series converges if and diverges if .
* Let be a series of real numbers. If ther series converges, then so does .
* Rearrangement stuff is cool, but unnecessary for this study guide.

### Gordon Chapter 7 (Sequences and Series of Functions)

* Let be a sequence of functions defined on an interval and let be a function defined on . The sequence **converges pointwise** to on if the sequence converges to for each . In other words, for all .
* Let be a sequence of functions defined on an interval and let be a function defined on . The series **converges pointwise** to on if the sequence of partial sums converges pointwise to on .
* Let be a sequence of functions defined on an interval and let be a function defined on . The sequence **converges uniformly** to on if for all there exists a positive integer such that for all and for all .
* A lot more information is here, may add later. I just don’t think it will help much for the Qual

### Gordon Chapter 8 (Point-Set Topology)

* A point is an **interior point** of if there exists a positive number such that
* A point is an **isolated point** of if there exists a positive number such that
* A point is a **limit point** of if for each positive number , the set contains a point of other than
* The set is **open** if all of its points are interior points
* The set is **closed** if it contains all of its limit points
* Every open interval is an open set and every closed interval is a closed set
* Let be a set of real numbers. A collection of sets is an **open cover** of if each set in is open and is contained in the union of all the sets in . The open cover has a **finite subcover** if is contained in the union of a finite number of sets in
* A set is **compact** if every open cover of has a finite subcover
* A compact set is closed and bounded
* A closed subset of a compact set is compact
* A set of real numbers is compact iff it is closed and bounded
* More is in this section. Possibly going to add more, but I don’t find it necessary for the Qual.

## Graduate Notes/Royden Book

### Royden Chapter 1 (Sets, Sequences, and Functions)

* A nonepmty set of real numbers is said to be **bounded above** provided that there is a real number such that for all . is known as an upper bound for . We define bounded below similarly.
* **The Completeness Axiom**: Let be a nonempty set of real numbers that is bounded above. Then among the set of upper bounds for there is a smallest, or least, upper bound.
* The least upper bound of is called the **supremum** of and denoted by . We define the **infimum** similarly as the greatest lower bound and denote it by .
* **Triangle Inequality**:
* A set of real numbers is said to be **inductive** provided it contains and if the number , the number as well.
* Every nonempty set of natural numbers has a smallest member.
* **Archimedean Property**: For each pair of positive real numbers and , there is a natural number for which .
* A set of real numbers is said to be **dense** in provided between any two real numbers there lies a member of .
* The rational numbers are dense in .
* A set is said to be **finite** provided either it is empty or there is a natural number such that is equipotent to .
* We say that is **countably infinite** provided is equipotent to the set (the natural numbers). A set that is either finite or countably finite is said to be **countable**. A set that is not countable is **uncountable**.
* A subset of a countable set is countable.
* A nonempty set is countable iff it is the image of a function whose domain is a nonempty countable set.
* The union of countable sets is countable.
* A set of real numbers is called **open** provided for each , there is a for which the interval is contained in .
* The set of real numbers and the empty set are open; the intersection of any finite collection of open sets is open; and the union of any collection of open sets is open.
* Every nonempty open set is the disjoint union of a countable collection of open intervals.
* For a set of real numbers, a real number is called a **point of closure** of provided every open interval that contains also contains a point in . The collection of points of closure of is called the **closure** of .
* A set of real numbers is open iff its complement in is closed.
* A collection of sets is said to be a **cover** of a set provided . By a subcover of a cover of we mean a subcollection of the cover that itself also is a cover of . If each set in a cover is open, then we call an **open cover** of . If the cover contains only a finite number of sets, we call it a **finite cover**.
* Let be a closed and bounded set of real numbers. THen every open cover of has a finite subcover.
* We say that a countable collection of sets is **descending** or **nested** provided that for every natural number . It is said to be **ascending** provided for every natural number .
* **-algebra**: Given a set , a collection of subsets of is called a -algebra provided

– the empty set belongs to

– the complement in of a set in also belongs to

– the union of a countable collection of sets in also belongs to .

* Let be a collection of subsets of a set . Then the intersection of all -algebras of subsets of that contain is a -algebra that contains . Moreover, it is the smallest -algebra of subsets that contains in the sense that any -algebra that contains also contains .
* The collection of Borel sets of real numbers is the smallest -algebra of sets of real numbers that contains all of the open sets of real numbers. (every open set is a Borel set)

### Royden Chapter 2 (Lebesgue Measure)

* **The measure of an interval is its length**. Each nonempty interval is Lebesgue measurable and
* **Measure is translation invariant**. If is Lebesgue measurable and is any number, then the translate of by , , also is Lebesgue measurable and
* **Measure is countably additive over countable disjoint unions of sets**. If is a countable disjoint collection of Lebesgue measurable sets, then
* The **outer measure** of an interval is its length, it is translation invariant, however the outer measure is not finitely additive. Instead:
* Let be a nonempty interval of real numbers. For a set of real numbers, consider the countable collections of nonempty open, bounded intervals that cover , that is, collections for which . We define the **outer measure** of , , to be
* A measure is **monotone** if for all , then .
* A set is said to be **measurable** provided for any set that
* Any set of outer measure zero is measurable. In particular, any countable set is measurable.
* The union of a finite collection of measurable sets is measurable.
* The union of a countable collection of measurable sets is measurable.
* Every interval is measurable.
* The collection of measurable sets is a -algebra that contains the -algebra of Borel sets. Each interval, each open set, each closed set, and each clopen set is measurable.
* The translate of a measurable set is measurable.
* If is a measurable set of finite outer measure that is contained in , then
* and
* The restriction of the set function outer measure to the class of measurable sets is called **Lebesgue Measure**. It is denoted by , so that if is a measurable set, its Lebesgue measure will be , defined by
* The Lebesgue measure defined on the -algebra of Lebesgue measurable sets assigns length to any interval, is translation invariant, and is countably additive.
* **The Continuity of Measure**: Lebesgue measure possesses the following continuity properties:

1. If is an ascending collection of measurable sets, then,
2. If is a descending collection of measurable sets and , then

* For a measurable set , we say that a property holds **almost everywhere** on , or it holds for almost all , provided there is a subset of for which and the property holds for all ~ .
* Let be a countable collection of measurable sets for which . Then almost all belong to at most finitely many of the ’s.
* Let be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers for which the collection of translates of , , is disjoint. Then
* Any set of real numbers with positive outer measure contains a subset that fails to be measurable.
* There are disjoint sets of real numbers and for which
* The Cantor set is a closed, uncountable set of measure zero
* The Cantor-Lebesgue function is an increasing continuous function that maps onto . Its derivative exists on the open set , the complement in of the Cantor set on while
* There is a measurable set, a subset of the Cantor set, that is not a Borel set.

### Royden Chapter 3 (Lebesgue Measurable Functions)

* Let the function have a measurable domain . Then the following statements are equivalent:

1. For all , the set is measurable
2. For all , the set is measurable
3. For all , the set is measurable
4. For all , the set is measurable

* An extended real-valued function defined on is said to be **Lebesgue measurable**, or simply **measurable**, provided its domain is measurable and it satisfies one of the four above statements.
* Let the function be defined on a measurable set . Then is measurable iff for all open sets , the inverse image of under , , is measurable
* A real valued function that is continuous on its measurable domain is measurable
* A monotone function that is defined on an interval is measurable
* Let be an extended real-valued function . Then if is measurable on and a.e. on , then is measurable on . For a measurable subset of , is measurable on iff the restrictions of to and ~ are measurable.
* Let and be measurable functions on that are finite a.e. on . For any and , is measurable on and is measurable on .
* Let be a measurable real-valued function defined on and a continuous real-valued function on all of . Then the composition is a measurable function on
* For a sequence of functions with common domain , a function on and a subset of , we say that the sequence converges to pointwise on provided for all ; and the sequence converges to pointwise a.e. on provided it converges to pointwise on ~ where ; and the sequence converges to uniformly on provided for each , there is an index for which on for all .
* Let be a sequence of measurable functions on that converges pointwise a.e. on to the function . Then is measurable.
* If is any set, the **characteristic function** of , , is the function on defined by
* A real-valued function defined on a measurable set is called **simple** provided it is measurable and takes only a finite number of values
* Let be a measurable real-valued function on . Assume is bounded on , that is there exists an for which on . Then for all , there are simple functions and defined on which have the following approximation properties on :
* An extended real-valued function on a measurable set is measurable iff there is a sequence of simple functions on which converges pointwise on to and has the property that on for all . If is nonnegative, we may choose to be increasing
* **Egoroff’s Theorem**: Assume has finite measure. Let be a sequence of measurable functions on that converges pointiwse on to the real-valued function . THen for all , there is a closed set contained in for which uniformly on and ~
* Under the assumptions of Egoroff’s Thm, for all and , there is a measurable subset of and an index for which on for all and ~.
* Let be a simple function defined on . Then for each , there is a continuous function on and a closed set contained in for which on and ~
* Let be a real-valued measurable function on . Then for all , there is a continuous function on amd a closed set contained in for which on and ~

### Royden Chapter 4 (Integration)

* The **upper and lower sums** for with respect to a partition are
* where is the infimum on the given partition, and is the supremum
* The **lower and upper Riemann integrals** of over are defined by (respectively)
* where is a partition of .
* If the two above mentioned integrals are equal, then we say that is **Riemann integrable** over .
* For a simple function defined on a set og finite measure , we define the integral of over by
* where and each
* Let be a finite disjoint collection of measurable subsets of a set of finite meausre . For , let be a real number. If on , then
* **Linearity and Monotonicity of Integration**: Let and be simple functions defined on a set of finite measure . Then for any and ,
* Moreover, if on , then
* A bounded function ona domain of finite measure is said to be **Lebesgue integrable** over provided its upper and lower Lebesgue integrals over are equal. The common value of the upper and lower integrals is called the **Lebesgue integral**.
* Let be a bounded function defined on the closed, bounded interval . If is Riemann integrable over , then it is Lebesgue integrable over and the two integrals are equal.
* Let be a bounded measurable function on a set of finite meausre . Then is integrable over .
* Let and be bounded measurable functions on a set of finite measure . Then for any and ,
* Moreover, if on , then
* Let be a bounded measurable function on a set of finite measure . Suppose and are disjoint meausrable subsets of . Then
* Let be a bounded measurable function on a set of finite measure . Then,
* Let be a sequence of bounded measurable functions on a set of finite measure . If uniformly on , then
* **The Bounded Convergence Theorem**: Let be a sequence of measurable functions on a set of finite measure . Suppose is uniformly pointwise bounded on , that is, there exists a number for which on for all . If pointwise on , then
* **Chebychev’s Inequality**: Let be a nonnegative measurable function on . Then for any ,
* Let be a nonnegative measurable function on . Then iff a.e. on .
* **Linearity and Monotonicity** follow for nonnegative measurable functions.
* **Fatou’s Lemma**: Let be a sequence of nonnegative measurable functions on . If pointwise a.e. on , then
* **Monotone Convergence Theorem**: Let be an increasing sequence of nonnegative measurable functions on . If pointwise a.e. on , then
* A nonnegative measurable function on a measurable set is said to be **integrable** over provided
* Let the nonnegative function be integrable over . Then is finite a.e. on .
* Let be a measurable function on . Then and are integrable over iff is integrable over .
* A measurable function on is said to be **integrable** over provided . When this is so, we define the integral by
* Let be integrable over . THen is finite a.e. on and if such that .
* **The Integral Comparison Test**: Let be a measurable function on . Suppose there is a nonnegative function that is integrable over and dominates in the sense that on . Then is integrable over and
* If and are integrable functions on , then **linearity** and **monotonicity** follow.
* Let be integrable over . Assume and are disjoint measurable subsets of . Then
* **Dominated Convergence Theorem**: Let be a sequence of measurable functions on . Suppose there is an integrable function on and dominates on in the sense that on for all . If pointwise a.e. on , then is integrable over and .
* **General Dominated Convergence Theorem**: Let be a sequence of measurable functions on that converges pointwise a.e. on to . Suppose there is a sequence of nonnegative measurable functions on that converges pointwise a.e. on to and dominates on in the sense that on for all . If
* then,
* Let be a set of finite measure and . Then is the disjoint union of a finite collection of sets, each of which has measure less than .
* A family of measuable functions on is said to be **uniformly integrable** over provided for each , there is a such that for each , if is measurable and , then .
* Let be a finite collection of functions, each of which is integrable over . Then is uniformly integrable.
* **Vitali COnvergence Theorem**: Let be of finite measure. Suppose the sequence of functions is uniformly integrable over . If a.e. on , then is integrable over and
* Let be a bounded function on a set of finite measure . Then is Lebesgue integrable over iff it is measurable.
* Let be a bounded function on the closed, bounded interval of . THen is Riemann integrable over iff the set of points in at which fails to be continuous has measure zero.

### Royden Chapter 6 (Differentiation)

* Let be a monotone function on the open interval . Then is continuous except possibly at a countable number of points in .
* If the function is monotone on the open interval , then it is differentiable almost everywhere on
* Define the **variation** of with respect to (a partition) by , and the **total variation** of on by where is a partition on
* A real valued function on the closed and bounded interval is said to be of **bounded variation** on provided
* **Jordan’s Thm**: A function is of bounded variation on the closed, bounded interval iff it is the difference of two increasing functions on
* If the function is of bounded variation on the closed and bounded interval then it is differentiable almost everywhere on the open interval and is integrable over
* A real valued function on a closed and bounded interval is said to be **absolutely continuous** on provided for each there is a such that for every finite disjoint collection of open intervals in , if , then
* If the function is Lipschitz on a closed, bounded interval , then it is absolutely continuous on
* Let the function be absolutely continuous on the closed, bounded interval . Then is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation
* Let the function be absolutely continuous on the closed, bounded interval . Then is differentiable almost everywhere on , its derivative is integrable over and
* We call a function on a closed, bounded interval the **indefinite integral** of over provided that is Lebesgue integrable over and for all
* A function on a closed, bounded interval is absolutely continuous on iff it is an indefinite integral over
* Let the function be monotone on the closed, bounded interval . Then is absolutely continuous on iff
* Let be integrable over the closed, bounded interval . Then for almost all iff for all
* Let be integrable over the closed, bounded interval . Then for almost all

### Royden Chapter 7 ( Spaces)

* For most of this section, unless otherwise stated, define to be a measurable set of real numbers, and to be the collection of all measurable extended real-valued functions on that are finite a.e. on . Define and to be equivalent and iff for almost all .
* We call a function **essentially bounded** provided there is some called an **essential upper bound** for for which for almost all
* **functionals** are real-valued functions that have as their domain linear spaces of functions
* Let be a linear space. A real-valued functional on is called a **norm** provided for each and in , and each real number , and iff ,
* By a **normed linear space** we mean a linear space together with a norm. If is a linear space normed by we say that a function in is a **unit function** provided
* For any , , the function is a unit function: it is a scalar multiple of which we call the **normalization** of
* **The Normed Linear Space**
* **The Normed Linear Space** : For a function , define to be the infimum of the essential upper bounds for . We call the **essential supremum** of and claim that is a norm on
* is a norm and is called the **maximum norm**
* For a measurable set where and a function in , define
* The conjugate of a number is the number , which is the unique number for which
* Note, the conjugate of is defined to be and vice versa.
* **Young’s Inequality**: For , is the conjugate of and any two positive numbers and ,
* Let be a measurable set , and be the conjugate of . If belongs to and belongs to , then their product is integrable over and
* This is known as **Holder’s Inequality**.
* Let be a measurable set and . If the functions and belong to , then so does their sum and moreover,
* **Cauchy-Schwarz Inequality**: Let be a maeasurable set and and measurable functions on for which and are integrable over . Then their product is also integrable over and
* Let be a measurable set and . Suppose is a family of functions in that is bounded in in the sense that there is a constant for which for all in . Then the family is uniformly integrable over .
* Let be a measurable set of finite measure and . Then . Furthermore for all in where if and if
* A sequence in a linear space that is normed by is said to **converge to in**  provided . This can also be written as in or in .
* A sequence in a linear space that is normed by is said to be **Cauchy** in provided for each , there is a natural number such that for all .
* A normed linear space is said to be **complete** provided every Cauchy sequence in converges to a function in . A complete normed linear space is called a **Banach space**
* Let be a normed linear space. Then every convergent sequence in is Cauchy. Moreover, a Cauchy sequence in converges if it has a convergent subsequence.
* Let be a linear space normed by . A sequence in is said to be **rapidly Cauchy** provided there is a convergent series of positive numbers for which for all
* Let be a normed linear space. Then every rapidly Cauchy sequence in is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.
* Let be a measurable set and . Then every rapidly Cauchy sequence in converges both wrt the norm and pointwise a.e. on to a function in .
* Let be a measurable set and . Then is a Banach space. Moreover, if in , a subsequence of converges pointwise a.e. on to
* Let be a measurable set and . Suppose is a sequence in that converges pointwise a.e. on to the function which belongs to . Then in iff
* Let be a measurable set and . Suppose is a sequence in that converges pointwise a.e. on to the function which belongs to . Then in iff is uniformly integrable and tight over .
* Let be a normed linear space with norm . Given two subsets and of with , we say that is **dense** in , provided for each function and , there is a function for which
* Let be a measurable set and . Then the subspace of simple functions in is dense in
* Let be a closed, bounded interval and . Then the subspace of step functions on is dense in
* A normed linear space is said to be **separable** provided there is a countable subset that is dense in .
* Let be a measurable set and . Then the normed linear space is separable.

### Royden Chapter 8 ( Spaces Continued)

* A **linear functional** on a linear space is a real-valued function on suxh that for and in and and real numbers,
* For a normed linear space , a linear functional on is said to be **bounded** provided there is an for which for all . The infimum of all such is called the **norm** of and denoted by
* Let be a normed linear space. Then the collection of bounded linear functionals on is a linear space on which is a norm. This normed linear space is called the **dual space** of and denoted by
* Let be a measurable set, , be the conjugate of , and belong to . Define the functional on by for all . Then is a bounded linear functional on and
* Let and be bounded linear functionals on a normed linear space . If on a dense subset of , then
* Let be a closed, bounded interval and . Suppose us a bounded linear functional on . Then there is a function in , where is the conjugate of for which for all
* Let be a measurable set, and the conjugate of . For each , define the bounded linear functional on by for all in . Then for each bounded linear functional on , there is a unique function for which and
* Let be a normed linear space. A sequence in is said to **converge weakly** in to in provided for all
* Let be a measurable set, , and the conjugate of . Then converges weakly in to in iff for all
* Let be a measurable set and . Suppose converges weakly in to . Then is bounded in and
* Let be a measurable set, , and the conjugate of . Suppose converges weakly to in and converges strongly to in . Then
* The **linear span** of a subset of a linear space is the linear space consisting of all linear combinations of functions in , that is, the linear space of functions of the form where each is a real number and each belongs to
* Let be a measurable set and . Suppose is a bounded sequence in and belongs to . Then converges weakly to in iff for every measurable subset of , . If , it is sufficient to consider sets of finite measure.
* Let be a closed and bounded interval and . Suppose is a bounded sequence in and belongs to . Then converges weakly to in iff
* for all . This theorem is false for
* Let be a measurable set and . Suppose converges weakly to in . Then in iff
* Let be a measurable set and . Suppose converges weakly in . Then a subsequence of converges strongly in to iff
* Let be a measurable set and . Then every bounded sequence in has a subsequence that converges weakly in to a function in